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Packing in trees

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Abstract

Let G be a graph and let v be a vertex of G. The open neighbourhood $N(v)$ of v is the set of all vertices adjacent with v in G, while the closed neighbourhood of v is $N(v) \cup \{v\}$. A *packing* of a graph G is a set of vertices whose closed neighbourhoods are pairwise disjoint. Equivalently, a *packing* of a graph G is a set of vertices whose elements are pairwise at distance at least 3 apart in G. The *lower packing number* of G, denoted $\rho_L(G)$, is the minimum cardinality of a maximal packing of G while the *(upper) packing number* of G, denoted $\rho(G)$, is the maximum cardinality among all packings of G. An *open packing* of G is a set of vertices whose open neighbourhoods are pairwise disjoint. The lower open packing number of G, denoted $\rho_L^o(G)$, is the minimum cardinality of a maximal open packing of G while the (upper) open packing number of G, denoted $\rho^o(G)$, is the maximum cardinality among all open packings of G. We present upper bounds on the packing number and the lower packing number of a tree. Bounds relating the packing numbers and open packing numbers of a tree are established. © 1998 Elsevier Science B.V. All rights reserved

Keywords: Bounds; Open packing; Packing; Trees

1. Introduction

In this paper, we follow the notation of [2]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighbourhood of $v \in V$ is $N(v)=\{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is $N[v]=\{v\} \cup N(v)$. If S is a subset of V, then the distance $d(v,S)$ from v to S is the minimum distance from v to a vertex of S. An end-vertex is a vertex of degree 1. We will refer to an end-vertex of a tree as a *leaf*.

For an integer $\ell \geqslant 1$, we define the *ℓ-corona* of a graph G to be the graph of order $(\ell+1)|V(G)|$ obtained from G by attaching a path of length ℓ to each vertex of G

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so that the resulting paths are vertex disjoint. The 1-corona of G is also called the corona of G.

A *packing* of a graph G is a set of vertices whose closed neighbourhoods are pairwise disjoint. Equivalently, a *packing* of a graph G is a set of vertices whose elements are pairwise at a distance at least 3 apart in G. The *lower packing number* of G, denoted $\rho_{\rm L}(G)$, is the minimum cardinality of a maximal packing of G while the (*upper*) *packing number* of G, denoted $\rho(G)$, is the maximum cardinality among all packings of G. The packing number of a graph has been studied in [1,3,7,8], and elsewhere.

A set S of vertices of G is an open packing of G if the open neighbourhoods of the vertices of S are pairwise disjoint in G. The *lower open packing number* of G, denoted $\rho_{\rm L}^{\rm o}(G)$, is the minimum cardinality of a maximal open packing of G while the (*upper*) *open packing number* of G, denoted $\rho^{\rm o}(G)$, is the maximum cardinality among all open packings of G. The open packing number of a graph has been studied in [6] for example.

In this paper, we investigate bounds on the packing number and the lower packing number of a tree. For a tree T, we investigate bounds relating $\rho(T)$ with each of the parameters $\rho_{\rm L}(T)$, $\rho^{\rm o}(T)$ and $\rho_{\rm L}^{\rm o}(T)$, and bounds relating $\rho_{\rm L}(T)$ with each of the parameters $\rho^{\rm o}(T)$ and $\rho_{\rm L}^{\rm o}(T)$.

2. Bounds on the packing numbers

We begin this section with the following upper bound on the packing number. The proof follows readily from the definition of a packing in a graph.

Theorem 1. *Let $G=(V,E)$ be a graph of order $n\geqslant 2$ with degree sequence $d_1,d_2,\ldots, d_n$ where $d_1\leqslant d_2\leqslant\cdots\leqslant d_n$. Then*

$$\rho(G)\leqslant \max\{k \mid k+d_1+\cdots+d_k\leqslant n\}.$$

If G is a graph of order n with minimum degree $\delta\geqslant 1$, then Theorem 1 implies that $\rho(G)\leqslant n/(\delta+1)$. This upper bound on the packing number of a graph in terms of its order and minimum degree is sharp for all $\delta\geqslant 1$. As a special case we have the following result due to Meir and Moon [7].

Theorem 2 (Meir and Moon [7]). *If T is a tree of order $n\geqslant 2$, then $\rho(T)\leqslant n/2$, and this bound is sharp. Furthermore, $\rho(T)=n/2$ if and only if T is the corona of a tree of order $n/2$.*

If G be a graph of order n with maximum degree $\Delta\geqslant 2$, then, using the definition of a maximal packing, it is easy to see that $\rho_{\rm L}(G)\geqslant n/(\Delta^2+1)$. This lower bound on the lower packing number of a graph in terms of its order and maximum degree is readily seen to be sharp. More interesting is to find a sharp upper bound on the lower packing number. We will restrict our attention to trees.

Since $\rho_L(G) \leqslant \rho(G)$ for all graphs G, we know that $\rho_L(T) \leqslant n/2$. This upper bound is asymptotically best possible; that is, for $m \geqslant 1$ an integer, there exist trees G_m of order n such that $\rho_L(G_m)/n \to \frac{1}{2}$ as $m \to \infty$ (take, for example, the tree G_m of Fig. 2 to be constructed later). It is our aim to show that this upper bound of $n/2$ can be improved. To do this, we introduce some additional notation.

For $m \geqslant 1$ an integer, if X and Y are subsets of vertices of a graph $G = (V, E)$, then the set X is defined in [4] to m-dominate Y if and only if each vertex of Y is within distance m from some vertex of X. In particular, if X m-dominates V, then X is called an *m-dominating set* of G. We will need the following result from [5].

Theorem 3 (Henning et al. [5]). *For $m \geqslant 1$, if $G = (V, E)$ is a connected graph of radius at least $m + 1$, then there exists a minimum m-dominating set $\mathscr{D}$ of G such that for each $v \in \mathscr{D}$, there exists a vertex $w \in V - \mathscr{D}$ at distance exactly m from v and at distance greater than m from every vertex of $\mathscr{D}$ different from v.*

If T is a rooted tree with root r and v is a vertex of T, then the *level number* of v, which we denote by $l(v)$, is the length of the unique r–v path in T. If a vertex v of T is adjacent to u and $l(u) > l(v)$, then u is called a *child* of v, and v is the *parent* of u.

We are now in a position to present the following sharp upper bound on the lower packing number of a tree.

Theorem 4. *If $T = (V, E)$ is a tree of order $n \geqslant 3$, then*

$$\rho_L(T) \leqslant \frac{n + 3 - 2\sqrt{n}}{2},$$

and this bound is sharp.

Proof. Let $\mathscr{D} = \{v_1, \ldots, v_b\}$ be a minimum 2-dominating set of T that satisfies the statement of Theorem 3 (with $m = 2$). We introduce the following notation. Let $\mathscr{I} = \{1, 2, \ldots, b\}$. For $i \in \mathscr{I}$, let

$W_i = \{w \in V - \mathscr{D} \mid d(v_i, w) = 2$ and w is at distance at least 3 from every vertex of $\mathscr{D}$ different from $v_i\}$,

$U_i = \{u \in V \mid u$ is adjacent to both v_i and w for some $w \in W_i\}$, and

$X_i = W_i \cup U_i \cup \{v_i\}$.

Thus, X_i consists of all vertices that belong to a v_i–w path of length 2 for some $w \in W_i$. By our choice of $\mathscr{D}$, we know that $W_i \neq \emptyset$ for all i. Hence $|X_i| \geqslant 3$ for all i. □

Claim 5. *$X_i \cap X_j = \emptyset$ for $1 \leqslant i < j \leqslant b$.*

Proof. Suppose $x \in X_i \cap X_j$ for some i and j with $1 \leqslant i < j \leqslant b$. Then there exists a vertex w_i (w_j) in W_i (W_j, respectively) such that the v_i–w_i path (v_j–w_j path,

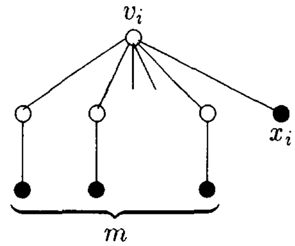

Fig. 1. The tree T_i wih $|S_i| = n_i/2$.

respectively) of length 2 contains the vertex x. But then at least one of w_i and w_j is within distance 2 from both v_i and v_j, which produces a contradiction. $\square$

By Claim 5, and since $\mathscr{D}$ 2-dominates V, we can partition V into sets $V_1, \ldots, V_b$, where each V_i induces a *tree* T_i of radius at most 2, and where $X_i \subseteq V_i$ and v_i 2-dominates V_i. For $i \in \mathscr{I}$, let $|V_i| = n_i$. Then $n_i = |V_i| \geqslant |X_i| \geqslant 3$ for all i. By the pigeonhole principle, at least one of the sets V_i contains at least n/b vertices. Relabelling the sets if necessary, we may assume that $n_1 \geqslant n/b$.

We now carefully construct a maximal packing S of T. To do this, we first construct a tree F as follows. For each subgraph T_i, $i \in \mathscr{I}$, we associate a vertex t_i. We now construct a tree F with vertex set $V(F) = \{t_1, t_2, \ldots, t_b\}$, where two vertices t_i and t_j are adjacent in F if and only if there is an edge joining a vertex of T_i and a vertex of T_j in T. Necessarily, F is tree.

We now root the tree F at the vertex t_1. Relabelling the vertices if necessary, we may assume that the vertices $t_1, t_2, \ldots, t_b$ are labelled in nondecreasing order of their level number from t_1 in F; that is, $\ell(t_i) \leqslant \ell(t_{i+1})$ for $i = 1, 2, \ldots, b-1$. Furthermore, we now let $\mathscr{I}_1 = \{i \mid i \in \mathscr{I} - \{1\}$ and T_i is the corona of a star on at least two vertices with v_i as its centre$\}$, and let $\mathscr{I}_2 = \mathscr{I} - \mathscr{I}_1$.

Suppose that S is any maximal packing of T containing v_1. For each $i \in \mathscr{I}$, let $S_i = S \cap V_i$. Then $S_1 = \{v_1\}$ and S_i is a packing in T_i. Hence, by Theorem 2, $|S_i| \leqslant n_i/2$. Since v_i 2-dominates V_i, it follows from Theorem 2 that $|S_i| = n_i/2$ if and if T_i is the corona of a star on at least two vertices with v_i as its centre. Hence, if $|S_i| = n_i/2$, then $i \in \mathscr{I}_1$. Furthermore, if $|S_i| = n_i/2$, then S_i consists of the $n_i/2$ leaves of T_i (see Fig. 1). It is our aim to avoid such sets S_i when constructing the set S.

We now construct a maximal packing S of T in such a way that for each $i \in \mathscr{I}_1$, $|S \cap V_i| < n_i/2$. To do this we initially set $S = \{v_1\}$. We then systematically examine each of the trees $T_2, T_3, \ldots, T_b$. At each stage, we carefully add at most one vertex from T_i to S so that any extension of S to a maximal packing of T will contain at most $(n_i - 1)/2$ vertices of T_i. More precisely, suppose we are currently examining the tree T_i. If $i \in \mathscr{I}_2$, then we add no vertex of T_i to S and proceed to examine the next tree T_{i+1} if $i < b$. On the other hand, suppose $i \in \mathscr{I}_1$. Let t_j be the parent of t_i in F. If S contains no vertex of T_j, then we add v_i to S. Otherwise, let s_j denote the vertex of T_j that belongs to S. If s_j is adjacent to a vertex of T_i, then we add no vertex of T_i to S.

If $d(s_j, V_i) = 2$, then we add v_i to S if $d(s_j, v_i) > 2$; otherwise, if $d(s_j, v_i) = 2$, then we know that $d(s_j, U_i) > 2$ and we add any vertex from U_i to S. Finally, if $d(s_j, V_i) > 2$, then we add v_i to S. Thereafter, we proceed to examine the next tree T_{i+1} if $i < b$. Once we have examined each of the trees $T_2, T_3, \ldots, T_b$, we then extend the set S constructed thus far to a maximal packing of T (by systematically adding vertices, if necessary, to S until we obtain a maximal packing in T). For each $i \in \mathscr{I}$, let $S_i = S \cap V_i$. Then $S_1 = \{v_1\}$ and S_i is a packing in T_i. Furthermore, by construction and by Theorem 2, it follows that

Claim 6. $|S_i| \leqslant (n_i - 1)/2$ *for all* $i \in \mathscr{I}$.

By Claim 6, and since $n_1 \geqslant n/b$, we therefore have

$$\begin{aligned}
\rho_L(T) \leqslant |S| &= |S_1| + \sum_{i=2}^{b} |S_i| \\
&\leqslant 1 + \sum_{i=2}^{b} (n_i - 1)/2 \\
&\leqslant 1 + ((n - n_1) - (b - 1))/2 \\
&\leqslant 1 + \left(n - \frac{n}{b} - b + 1\right)/2 \\
&= \frac{1}{2}\left(n + 3 - \frac{n}{b} - b\right).
\end{aligned}$$

The last expression is maximized with $b = \sqrt{n}$. Thus,

$$\rho_L(T) \leqslant |S| \leqslant (n + 3 - 2\sqrt{n})/2.$$

This completes the proof of the upper bound. That this upper bound is sharp, may be seen as follows. For $m \geqslant 2$ an integer, let T be the tree obtained from a star $K_{1,m}$ by subdividing each edge once. Let $T_1, T_2, \ldots, T_{2m+1}$ be $2m+1$ disjoint copies of T, and let v_i denote the central vertex of T_i for $i = 1, 2, \ldots, 2m+1$. Finally, let G_m be the tree obtained from the disjoint union $\bigcup_{i=1}^{2m+1} T_i$ of $T_1, T_2, \ldots, T_{2m+1}$ by adding the edges $v_1 v_i$ for $i = 2, \ldots, 2m+1$. The tree G_m is shown in Fig. 2.

We show that $\rho_L(G_m) = 2m^2 + 1$. Let S be a maximal packing of G_m. Then S contains at most one of $v_1, v_2, \ldots, v_{2m+1}$. If $v_i \notin S$, then $|S \cap V(T_i)| = m$, while if $v_i \in S$, then $|S \cap V(T_i)| = 1$. Thus, $\rho_L(G_m) \geqslant 2m^2 + 1$. However, there exists a maximal packing of G_m of cardinality $2m^2 + 1$ as illustrated by the set of darkened vertices in Fig. 2, so $\rho_L(G_m) \leqslant 2m^2 + 1$. Consequently, G_m is a tree of order $n = (2m+1)^2$ with $\rho_L(G_m) = 2m^2 + 1$. Thus,

$$\rho_L(G_m) = 2m^2 + 1 = \frac{n + 3 - 2\sqrt{n}}{2}. \qquad \square$$

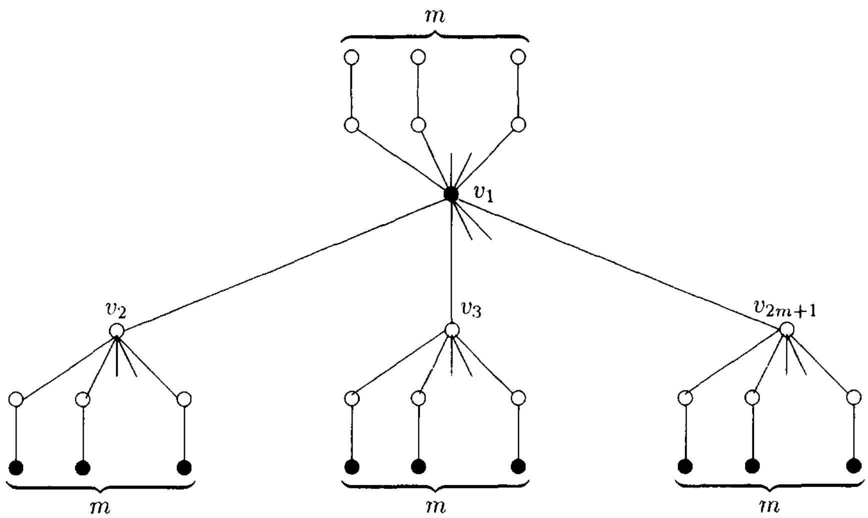

Fig. 2. The tree G_m. (The darkened vertices form a maximal packing in G_m.)

Table 1

	Upper bound	Family of trees attaining upper bound
ρ^o	2ρ	2-corona of a tree
ρ	ρ^o	corona of a tree
ρ_L^o	$2\rho_L$	3-corona of a tree
ρ^o	$2n/3$	2-corona of a tree
ρ	$n/2$	Corona of a tree
ρ_L	$\frac{1}{2}(n+3-2\sqrt{n})$	G_m (see Fig. 2)

We close this section with the following results, the proofs of which are straightforward and therefore omitted. The packing number of a graph is at most the open packing number, while the open packing number is at most twice the packing number and the lower open packing number is at most twice the lower packing number. Table 1 summarizes those results which establish upper bounds on packing parameters in trees of order n. (The upper bound of $2n/3$ on ρ^o was established in [6].)

3. Bounds relating packing parameters in trees

In this section we investigate bounds relating the packing parameters in trees. We begin by defining two families of trees. Let $\mathscr{F}_1$ denote the family of all trees obtained from a star $K_{1,m}$, $m \geqslant 1$, by subdividing each edge exactly once. Let T be obtained from a star $K_{1,m+1}$, $m \geqslant 1$, by subdiving each edge exactly once, and let T_1 and T_2 be two disjoint copies of T. Let F be the tree obtained from the disjoint union $T_1 \cup T_2$

Table 2

	Upper bound	Family of trees attaining upper bound
$\rho^{o} - \rho^{o}_{L}$	$(n-2)/2$	$\mathscr{F}_2$
$\rho - \rho_{L}$	$(n-2)/2$	Corona of a star $K_{1,m}$
$\rho^{o} - \rho$	$n/3$	2-corona of a tree
$\rho^{o} - \rho_{L}$	$(n-1)/2$	$\mathscr{F}_1$
$\rho - \rho^{o}_{L}$	$(n-4)/2$	Corona of a star $K_{1,m}$
$\rho^{o}_{L} - \rho$	$n/3$	Asymptotically best possible
$\rho^{o}_{L} - \rho_{L}$	$n/3$	Asymptotically best possible

of T_1 and T_2 by joining their central vertices with an edge. Let $\mathscr{F}_2$ denote the family of all such trees F.

Table 2 summarizes those results which establish upper bounds on the difference between certain packing parameters in terms of the order n of a tree. (The upper bound of $(n-2)/2$ on $\rho^{o} - \rho^{o}_{L}$ was established in [6].)

If $F \in \mathscr{F}_2$, then $\rho^{o}_{L}(F) = 2$ while $\rho_{L}(F) = m+2$. Thus $\rho_{L}(F) - \rho^{o}_{L}(F) = m$. Hence, the lower packing number of a tree can exceed its lower open packing number by an arbitarily large amount.

We present the proofs of two of the results stated in Table 2. The results in Table 2 which are not proved here are either similar to the given proofs or are straightforward to verify and left for the reader.

Theorem 7. *If T is a tree of order $n \geqslant 3$, then $\rho^{o}(T) - \rho(T) \leqslant n/3$.*

Proof. We proceed by induction on the order $n \geqslant 3$ of a tree. If T is a tree of order $n \leqslant 5$ that is not a path on five vertices, then $\rho^{o}(T) = 2$ while $\rho(T) \geqslant 1$, and the result is immediate. On the other hand, if T is a path on $n = 5$ vertices, then $\rho^{o}(T) = 3 = \rho(T) + 1$, and once again the result follows. Hence, the result is true for all trees of order 3,4 or 5. So, assume that for all trees T' of order $n' \geqslant 3$ where $n' < n$ and $n \geqslant 6$, that $\rho^{o}(T') - \rho(T') \leqslant n'/3$. Let T be a rooted tree of order n. We show that $\rho^{o}(T) - \rho(T) \leqslant n/3$.

Let w be a leaf of T at furthest distance from the root (so w is a vertex of T with maximum level number), and let v be the parent of w.

If T contains a vertex adjacent with at least two leaves, then removing one of these leaves produces a tree T' of order $n' = n-1$ satisfying $\rho^{o}(T') = \rho^{o}(T)$ and $\rho(T') = \rho(T)$. Thus, applying the inductive hypothesis, we have $\rho^{o}(T) - \rho(T) = \rho^{o}(T') - \rho(T') \leqslant n'/3 < n/3$. Hence, we may assume that every vertex of T is adjacent with at most one leaf. In particular, v has degree 2. Let u be the parent of v in T.

If u has degree 2, then let x be the parent of u and consider the nontrivial tree $T' = T - \{u, v, w\}$ of order $n' = n-3 \geqslant 3$. Since every maximal open packing of T contains two of the vertices u, v, w, x, we may assume, without loss of generality, that there is a maximum open packing S of T containing v and w. Hence, $S - \{v, w\}$ is an open packing of T', so $\rho^{\circ}(T') \geqslant \rho^{\circ}(T) - 2$; equivalently, $\rho^{\circ}(T) \leqslant \rho^{\circ}(T') + 2$. On the other hand, every maximal packing of T' can be extended to a maximal packing of T by adding the vertex w, so $\rho(T) \geqslant \rho(T') + 1$. Thus, applying the inductive hypothesis, we have $\rho^{\circ}(T) - \rho(T) \leqslant \rho^{\circ}(T') - \rho(T') + 1 \leqslant n'/3 + 1 = (n-3)/3 + 1 = n/3$. Hence, we may assume that u has degree $k+1 \geqslant 3$.

Any maximal open packing of T contains one vertex in $N(u)$ and contains either u or w. We may assume, without loss of generality, that there is a maximum open packing of T containing v and w. Furthermore, any maximal packing of T contains one vertex in $N[v]$. So we may assume that there is a maximum packing of T containing w. It follows that if u is adjacent with a leaf y, then the tree $T' = T - y$ of order $n' = n - 1$ satisfies $\rho^{\circ}(T') = \rho^{\circ}(T)$ and $\rho(T') \leqslant \rho(T)$. Thus, applying the inductive hypothesis, we have $\rho^{\circ}(T) - \rho(T) \leqslant \rho^{\circ}(T') - \rho(T') \leqslant n'/3 < n/3$. Hence, we may assume that every child of u has degree 2. Thus, the maximal subtree of T rooted at u is isomorphic to $K_{1,k}$ with each edge subdivided once. Let $v_1, \ldots, v_k$ be the children of u, and let w_i be the leaf adjacent with v_i, $1 \leqslant i \leqslant k$, where $v = v_1$ and $w = w_1$.

We now consider the nontrivial tree $T' = T - \{v_1, w_1\}$ of order $n' = n - 2$. Every maximal open packing of T contains at most one child of u, so we may assume, without loss of generality, that there is a maximum open packing S of T that does not contain v_1. If $u \in S$, then S contains none of the leaves $w_1, w_2, \ldots, w_k$. But then $(S - \{u\}) \cup \{w_1, w_2, \ldots, w_k\}$ would be an open packing of T of cardinality exceeding that of S, producing a contradiction. Thus $u \notin S$. Consequently, $\{w_1, w_2, \ldots, w_k\} \subset S$. Hence $S - \{w_1\}$ is a maximal open packing of T', so $\rho^{\circ}(T') \geqslant \rho^{\circ}(T) - 1$; equivalently, $\rho^{\circ}(T) \leqslant \rho^{\circ}(T') + 1$. On the other hand, a maximum packing of T' contains at most one of u, v_2 and w_2. So we may assume that T' has a maximum packing S' with $w_2 \in S'$ (so $u \notin S'$). Thus $\rho(T) \geqslant |S' \cup \{w_1\}| = \rho(T') + 1$. Hence, applying the inductive hypothesis, we have $\rho^{\circ}(T) - \rho(T) \leqslant (\rho^{\circ}(T') + 1) - (\rho(T') + 1) = \rho^{\circ}(T') - \rho(T') \leqslant n'/3 < n/3$. This completes the inductive proof. □

Theorem 8. *If T is a tree of order $n \geqslant 3$, then $\rho^{\circ}(T) - \rho_{\mathrm{L}}(T) \leqslant (n-1)/2$.*

Proof. We proceed by induction on the order $n \geqslant 3$ of a tree. If T is a tree of order $n \leqslant 5$ that is not a path on five vertices, then $\rho^{\circ}(T) = 2$ while $\rho_{\mathrm{L}}(T) = 1$, while if T is a path on $n = 5$ vertices, then $\rho^{\circ}(T) = 3 = \rho_{\mathrm{L}}(T) + 2$. If T is a tree of order $n = 6$ that is not a path on six vertices, then $\rho^{\circ}(T) \leqslant 3$ while $\rho_{\mathrm{L}}(T) = 1$, while if T is a path on $n = 6$ vertices, then $\rho^{\circ}(T) = 4 = \rho_{\mathrm{L}}(T) + 2$. Hence, for all trees of order $n \leqslant 6$ the result is readily checked to be true. So, assume that for all trees T' of order $n' \geqslant 3$ where $n' < n$ and $n \geqslant 7$, that $\rho^{\circ}(T') - \rho_{\mathrm{L}}(T') \leqslant (n'-1)/2$. Let T be a rooted tree of order n. We show that $\rho^{\circ}(T) - \rho_{\mathrm{L}}(T) \leqslant (n-1)/2$.

Let w be a leaf of T at furthest distance from the root, and let v be the parent of w. If T contains a vertex adjacent with at least two leaves, then removing one of these leaves produces a tree T' of order $n' = n - 1$ satisfying $\rho^{\circ}(T') = \rho^{\circ}(T)$ and $\rho_{\mathrm{L}}(T') = \rho_{\mathrm{L}}(T)$. Thus, applying the inductive hypothesis, we have $\rho^{\circ}(T) - \rho_{\mathrm{L}}(T) =$

$\rho^{\mathrm{o}}(T') - \rho_{\mathrm{L}}(T') \leqslant (n' - 1)/2 < (n - 1)/2$. Hence, we may assume that every vertex of T is adjacent with at most one leaf. In particular, v has degree 2. Let u be the parent of v in T. We consider two cases.

Case 1: $\deg u = k + 1 \geqslant 3$. Suppose that u is adjacent with a leaf y. Consider the tree $T' = T - y$ of order $n' = n - 1$. Any maximal open packing of T contains one vertex in $N(u)$ and contains either u or w. We may assume, without loss of generality, that there is a maximum open packing S of T containing v and w. Thus, S is also an open packing of T', so $\rho^{\mathrm{o}}(T') \geqslant \rho^{\mathrm{o}}(T)$. We show next that $\rho_{\mathrm{L}}(T') \leqslant \rho_{\mathrm{L}}(T)$. Let S^* be a maximal packing of T of minimum cardinality. Then S^* contains one vertex in $N[v]$. If $w \in S^*$, then $y \notin S^*$, for otherwise $(S^* - \{w, y\}) \cup \{v\}$ would be a maximal packing of T of cardinality less than that of S^*. Hence $w \notin S^*$. Thus either u or v belongs to S^*. In either event, S^* is a maximal packing of T'. Hence $\rho_{\mathrm{L}}(T') \leqslant |S^*| = \rho_{\mathrm{L}}(T)$. Thus, applying the inductive hypothesis, we have $\rho^{\mathrm{o}}(T) - \rho_{\mathrm{L}}(T) \leqslant \rho^{\mathrm{o}}(T') - \rho_{\mathrm{L}}(T') \leqslant (n' - 1)/2 < (n - 1)/2$.

Hence, we may assume that every child of u has degree 2. Thus, the maximal subtree of T rooted at u is isomorphic to $K_{1,k}$ with each edge subdivided once. Let $v_1, \ldots, v_k$ be the children of u, and let w_i be the leaf adjacent with v_i, $1 \leqslant i \leqslant k$, where $v = v_1$ and $w = w_1$.

We now consider the nontrivial tree $T' = T - \{v_1, w_1\}$ of order $n' = n - 2$. As in the proof of Theorem 7, $\rho^{\mathrm{o}}(T) \leqslant \rho^{\mathrm{o}}(T') + 1$. On the other hand, let S^* be a maximal packing of T of minimum cardinality. If $u \in S^*$, then S^* is also a maximal packing of T', so $\rho_{\mathrm{L}}(T') \leqslant |S^*| = \rho_{\mathrm{L}}(T)$. If $u \notin S^*$, then we may assume that $v_1 \notin S^*$ (if $v_1 \in S^*$, then we may replace v_1 and w_2 in S^* with w_1 and v_2). Thus, $w_1 \in S^*$ and $S^* - \{w_1\}$ is a maximal packing of T', whence $\rho_{\mathrm{L}}(T') \leqslant |S^*| - 1 = \rho_{\mathrm{L}}(T) - 1$, so certainly $\rho_{\mathrm{L}}(T) \geqslant \rho_{\mathrm{L}}(T')$. Hence, applying the inductive hypothesis, we have $\rho^{\mathrm{o}}(T) - \rho_{\mathrm{L}}(T) \leqslant (\rho^{\mathrm{o}}(T') + 1) - \rho_{\mathrm{L}}(T') \leqslant (n' - 1)/2 + 1 = (n - 1)/2$.

Case 2: $\deg u = 2$. Let x be the parent of u. Suppose that x is adjacent with a leaf y. Consider the tree $T' = T - \{w, y\}$ of order $n' = n - 2 \geqslant 5$. We may then assume, without loss of generality, that there is a maximum open packing S of T containing v, w and y. Thus $(S - \{w, y\}) \cup \{u\}$ is a maximal open packing of T', so $\rho^{\mathrm{o}}(T') \geqslant \rho^{\mathrm{o}}(T) - 1$; equivalently, $\rho^{\mathrm{o}}(T) \leqslant \rho^{\mathrm{o}}(T') + 1$. Morover, $\rho_{\mathrm{L}}(T') \leqslant \rho_{\mathrm{L}}(T)$. Thus, applying the inductive hypothesis, we have $\rho^{\mathrm{o}}(T) - \rho_{\mathrm{L}}(T) \leqslant \rho^{\mathrm{o}}(T') - \rho_{\mathrm{L}}(T') + 1 \leqslant (n' - 1)/2 + 1 = (n - 1)/2$. Hence, we may assume that x is adjacent to no leaf.

Suppose that there is a leaf w', different from w, at distance 3 from x. Let x, u', v', w' denote the x–w' path. We know that v' has degree 2. Further, we may assume that u' has degree 2, for otherwise we have Case 1. We now consider the tree $T' = T - \{u, v, w\}$ of order $n' = n - 3 \geqslant 4$. We may assume there is a maximum open packing of T containing v and w, so $\rho^{\mathrm{o}}(T) \leqslant \rho^{\mathrm{o}}(T') + 2$. Let S be a maximal packing of T of minimum cardinality, and let $S' = S \cap V(T')$. Then S contains one vertex in $N[v]$ and one vertex in $N[v']$, so we may assume that $u \notin S$. Consequently, S' is a maximal packing of T', whence $\rho_{\mathrm{L}}(T') \leqslant |S'| = |S| - 1 = \rho_{\mathrm{L}}(T) - 1$. Thus, applying the inductive hypothesis, we have $\rho^{\mathrm{o}}(T) - \rho_{\mathrm{L}}(T) \leqslant (\rho^{\mathrm{o}}(T') + 2) - (\rho_{\mathrm{L}}(T') + 1) \leqslant (n' - 1)/2 + 1 = (n - 4)/2 + 1 < (n - 1)/2$.

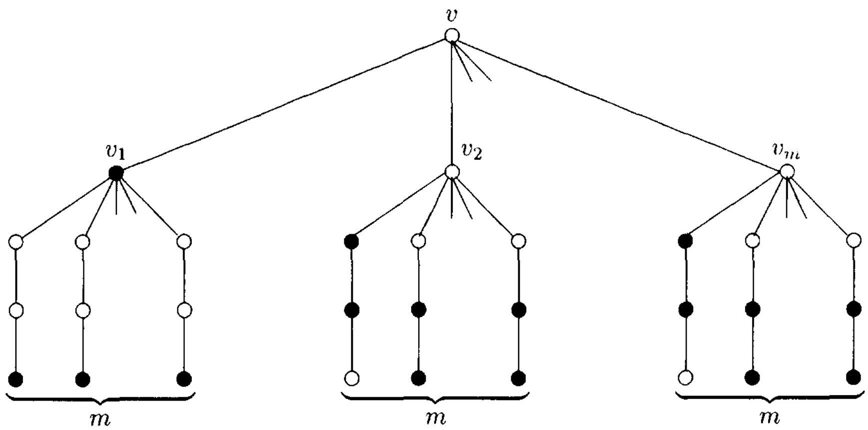

Fig. 3. The tree H_m. (The darkened vertices form a maximal open packing in H_m.)

Suppose that there is a leaf at distance 2 from x. Once again, applying the inductive hypothesis to the tree $T' = T - \{u, v, w\}$, we may show that $\rho^o(T) - \rho_L(T) \leqslant (n-1)/2$.

Finally, suppose that x has degree 2. We now consider the tree $T' = T - \{u, v, w, x\}$ of order $n' = n - 4 \geqslant 3$. Then $\rho^o(T) \leqslant \rho^o(T') + 2$ and $\rho_L(T') \leqslant \rho_L(T)$. Thus, applying the inductive hypothesis, we have $\rho^o(T) - \rho_L(T) \leqslant \rho^o(T') - \rho_L(T') + 2 \leqslant (n'-1)/2 + 2 = (n-1)/2$. This completes the inductive proof.

If $T \in \mathscr{F}_1$, then T is a tree of order $n = 2m+1$ satisfying $\rho^o(T) = m+1$ and $\rho_L(T) = 1$. Thus, $\rho^o(T) - \rho_L(T) = m = (n-1)/2$. Hence, this upper bound is sharp. □

Using similar inductive proofs to those employed in Theorems 7 and 8 one may establish that if T is a tree of order $n \geqslant 3$, then $\rho_L^o(T) - \rho_L(T) \leqslant n/3$. That this bound is best possible may be seen by considering the tree H_m constructed as follows. For $m \geqslant 2$ an integer, let T be the tree obtained from a star $K_{1,m}$ by subdividing each edge twice. Let $T_1, T_2, \ldots, T_m$ be m disjoint copies of T, and let v_i denote the central vertex of T_i for $i = 1, 2, \ldots, m$. Finally, let H_m be the tree obtained from the disjoint union $\bigcup_{i=1}^{m} T_i$ of $T_1, T_2, \ldots, T_m$ by adding a new vertex v and the edges vv_i for $i = 1, 2, \ldots, m$. The tree H_m is shown in Fig. 3.

The tree H_m has order $n = 3m^2 + m + 1$ and satisfies $\rho_L^o(H_m) = 2m^2 - m + 1$, $\rho(H_m) = m^2 + 1$ and $\rho_L(H_m) = m^2$. Hence,

$$\frac{\rho_L^o(H_m) - \rho_L(H_m)}{n} = \frac{m^2 - m + 1}{3m^2 + m + 1} = \frac{1 - 1/m + 1/m^2}{3 + 1/m + 1/m^2}.$$

Therefore, $(\rho_L^o(H_m) - \rho_L(H_m))/n \to \frac{1}{3}$ as $m \to \infty$. The tree H_m also illustrates that the bound $\rho_L^o(T) - \rho(T) \leqslant n/3$ is best possible.

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